FLATLAND,

by Edwin Abbot (1880's);

A Romance of Many Dimensions,

SPHERELAND,

by Dionys Burger (1965)

A Fantasy About Curved Spaces & an Expanding Universe

EINSTEIN HAPPENED

by Isaac Asimov

Why was it necessary to write a sequel to *Flatland*?

Flatland, published originally about 1880, is a charming geometric fantasy which, in the guise of dealing with living, thinking creatures introduces the reader, painlessly, to the mysteries of dimensional thinking. —Even to that all-but-ungraspable concept, the fourth dimension.

But surely geometric fantasies are not subject to obsolescence. Triangles, squares and spheres are today what they were in 1880 and will remain so indefinitely into the future. Why, then, continue the story? What more is there to say?

For one thing, the original *Flatland* was not about mathematics only. It also described a society in which casual assumptions were made concerning class distinctions and, in particular, women. The Victorian convention of women as a quite inferior form of life was accepted without question.

In the twentieth century, this view, both insulting and injurious, to say nothing of being untrue, could not be accepted, and Dionys Burger, the Dutch mathematician who published *Sphereland* in 1960, went to some pains to neutralize it. (Good for him!)

But there was more. Even the mathematics needed additions, for though a square remains a square, subtle changes in our understanding of what squares may be like take place as we gain a better understanding of the Universe.

Let me give you an example. The Earth we live on, in general, looks flat. It is lumpy and bumpy, of course, with hills and ravines, but there doesn't seem to be any general slant. If the unevennesses are averaged out, everything would be flat. At least so our eyes tell us. And since the Earth *looks* flat. the ancients thought, with considerable justification, that it *was* flat.

Today, however, we know that the Earth is spherical. Its surface is curved. The Earth looks just as flat as ever. Nothing has changed as far as our eyes are concerned, but our *understanding* of the Earth as a whole has changed. We know that the surface is curved, but so gently, so unnoticeably, that it still looks flat. Nevertheless, we can't say it *is* flat, because if we travel large distances then a guiding map drawn on the assumption that the Earth is flat will mislead us. Only a globe, or a map that is flat but takes the globularity of the Earth into account will lead us aright.

Well, the Universe isn't flat, either. If it were flat, then a ray of light traveling through a vacuum would move in a perfectly straight path. It would be as close to a perfectly straight line as we can imagine. And if we caused such a ray of light to produce a visible path (as by making it travel through foggy air that would scatter its light slightly all along its progress), that path would certainly *look* straight, just as the Earth certainly *looks* flat.

Just the same the path of the ray of light is not *perfectly* flat. It deviates slightly, *very* slightly, from the straight line. It deviates far less than the surface of the Earth deviates from flatness.

Of course, the deviation builds up. If we were to imagine the path of the ray of light made very long indeed—from here to some distant star, for instance—then it would be quite clear that it did not follow a perfectly straight line. A map of the Universe based on light traveling in a perfectly straight line would not be accurate and if we tried to use it as a guide to travel great distances, we would find ourselves led totally astray—just as we would go astray if we tried to use a map of a flat Earth in order to travel from the United States to New Zealand.

Who told us that the Universe was not flat, but curved?

It was a German-born scientist, Albert Einstein, Between 1905 and 1916, he worked out a totally new way of looking at the Universe, a way that, at first glance, seemed very complicated and "against common sense." Part of it was that light did not travel in straight lines, but followed paths that had very slight curves to them.

Just the same, Einstein's view, however nonsensical it may have seemed to people unprepared for it, made interesting predictions that turned out to be true—that it was impossible to go faster than light in a vacuum, that certain strange changes took place as you approached the speed of light, that mass and energy could be interchanged, and so on.

Of course, it was difficult to accept this, but that's not surprising. Imagine how difficult it must have been for people to believe the Earth was a sphere when they could "see with their own eyes" that it was flat.

When we think of a square ordinarily, we think of it as bounded by four "straight lines." But those straight lines do not really exist in our Universe. When we draw a straight line, it is

actually very slightly curved, because it follows the curve of the Universe just as a ray of light does. The curve is so slight that, under all ordinary conditions we can ignore it, but there are scientific situations today in which we have to take such curvatures into account, or we will never understand why the Universe behaves as it does.

That is why it was necessary to write *Sphereland* as a sequel to *Flatland*. In between *Flatland* in 1880, and *Sphereland* in 1960, Einstein happened, and suddenly we were forced to realize that all those straight lines we have always dealt with so confidently were a little more complicated than we thought.

Sphereland, in its way, then, is a geometric introduction to the Einsteinian Universe.

Fear not, however. It contains no difficult mathematics and it won't sprain your understanding. It remains just what *Flatland* was to begin with—a pleasant fantasy. You will have no sensation of "learning" whatsoever, but you will learn just the same. You won't be able to help it. And when you are done with *Sphereland* you will find that someday, if you should happen to encounter the Einsteinian view, it will be that much less difficult to grasp, thanks to what you have picked up in this book.

Preface

[which does not need to be read]

About 1880 a book appeared which immediately caught on like wildfire. Its title was *Flatland, a Romance of Many Dimensions, by A Square*, and it was written by Dr. Edwin A. Abbott (born in 1838), an educator endowed with exceptional qualities who became headmaster of a London school in 1865. In addition to a number of textbooks, he wrote some theological works, a biography of Bacon and a Shakespearean grammar which was valued highly. With the appearance of *Flatland*, however, he gained a widespread reputation that has remained firmly established for many decades and may well continue to do so for centuries.

One of the book's outstanding qualities is its educational value — that is, it tells the layman about a difficult subject in a very informal and easy-paced way so that he can grasp the subject in ways an ordinary textbook could not provide. Not only "outsiders," however, but insiders, scientists, have enjoyed this fantasy and still do, even though the style has become rather old-fashioned.

In *Flatland* the difficulties of forming any idea of the "fourth dimension" are portrayed through the problems that a Square, an inhabitant of a two-dimensional world, would have in imagining a third direction perpendicular to his two dimensions.

Following this same line, the author of *Sphereland* tells of the grandson of the Square, a Hexagon, who is confronted by even greater problems that can be understood only by assuming that the plane he lives in is curved and, even more confounding, that his two-dimensional world is expanding. It would be quite difficult to explain such phenomena with ease in the framework of a nonfiction "popular" science book, and it is clear that the story form has simplified the task greatly. But *Sphereland*, like *Flatland*, is also intended for the scientist, who, we hope, will derive much enjoyment from the story and its characters.

Since this book is a sequel to Abbott's fantasy, it starts off with a summary of *Flatland* which will make many readers want to read that book for themselves. Fortunately, it is available. In America editions are published by Barnes & Noble, Inc., and Dover Publications, Inc., New York and in England, by Basil Blackwell of Oxford. *Flatland* is certainly worth reading!

In Abbott's days the fourth dimension could only be imagined mathematically. Today, with the structure of our space the object of constant discussion, this dimension is of even more interest than it was in the nineteenth century. Abbott discussed only a higher-dimensional Euclidean space. We should be grateful that he did not include any discussion of non-Euclidean spaces: it would have made everything needlessly complicated. For the same reason the author of *Sphereland* deals only with curved spaces of regular positive curvature (spherical curvature) and does not mention negative or even more intricate curvatures.

It is possible that in describing his nonexistent fantasy worlds, the author has made errors about physical laws. The field of physics is developing very rapidly nowadays. The concept of an expanding universe whose three-dimensional space is the surface of an expanding hypersphere is already rather obsolete. However, if the author had tried to follow science's evolutionary course exactly, the main purpose of the book — to give the reader some insight into the principles of curved and expanding spaces — would have been defeated.

The idea that a three-dimensional man could be "reversed" by the Over-Sphere and returned to his own space without undergoing serious hardship should be taken with a grain of salt. Such a transformation would reverse all the stereoisomers, i.e., almost every chemical compound in his body, so that he would not be able to digest unreversed food and would soon starve to death. The author allows antimatter to exist, but in fact it is destroyed by matter (the effect is mutual). It is not known conclusively what antimatter is. It may be that it is "mirror-reversed" matter. If so, a reversed man would explode instantly.

We must not be so overscrupulous in writing a scientific fantasy as to do away with the entertainment value of the book. H. G. Wells and Jules Verne made many a mistake in their scientific stories, and yet, aren't their products still quite plausible? Thus, the author finds himself in good company.

Sphereland is written not only for the reader who wants to get an insight into difficult space problems, but also for

scientifically trained readers who will enjoy this fantasy novel at their own level.

If in reading this book the reader experiences only a small part of the joy the author felt in writing it, the latter will be quite satisfied.

DIONYS BURGER

Zeist, The Netherlands

SPHERELAND

A Look at FLATLAND *A Fantasy About the Fourth Dimension*

by A SQUARE

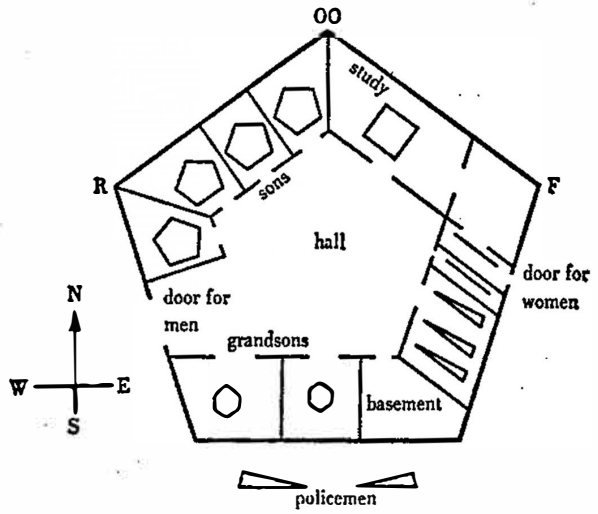
1 *Flatland and Its Inhabitants*

Visualize a flat extended plane in which two-dimensional geometric figures—somewhat like shadows, but hard and with bright, shiny borders—can move in all directions, this is Flatland and its inhabitants.

The country has a weak gravitational force in a direction called south. The opposite direction is considered to be north and in between, on either side, are east and west. That force enables the inhabitants to orientate themselves.

In certain moderate zones, however, the force is small. While the "delicate" female can sense it very easily, the cruder male occasionally has trouble with it. For example, someone traveling well outside of the inhabited areas may lose his sense of direction. If this should happen, he will have to wait for a rainstorm, since the rain always comes from the north.

The houses are built with this in mind. They are pentagonal. R OO F is the roof against the rain. On the left is



The house of the Square. The bedroom of the man of the house is next to the study. The woman of the house has her own entrance, the Door for Women. Next to it is the daughter's bedroom, and three servants are in the kitchen.

a wide door for men, on the right a narrow one for the women—who are much slimmer, as we shall see.

No windows are needed in private homes because there is light everywhere in Flatland. No one knows where it originates. Perhaps outside the plane, in the space of three dimensions? Square and triangular houses are not permitted because their sharp corners would be dangerous to passers-by. We must not forget that vision in Flatland is rather poor since its inhabitants never see more than a line and are barely able to distinguish an angle. The sides of inanimate objects give out very little light and are even more difficult

to see. Only military installations such as forts, powder magazines, barracks, and certain government buildings are allowed to have sharper corners, if they are located on grounds not open to the public.

The adult inhabitants are about 11 inches long. The women have the shape of a straight line of almost no width; the men are triangles or polygons. The least developed men are formed like very sharply pointed isosceles triangles with a base of not more than an eighth of an inch and sides of approximately 11 inches each. Consequently, their top angle is very small, and since it contains the brains (and is therefore also known as the "brain angle"), it's clear that we have to do here with the intellectually less gifted.

As each generation succeeds the last, the top angle increases, providing more space for brains. With good behavior (and not otherwise) each successive generation comes to have a top angle half a degree larger than the previous one. Misbehavior or worse can cause an individual to slide down again one or more steps on the social scale.

Whenever a vertical angle of 60° has been achieved, it is first verified by a Sanitary and Social Board established especially for the purpose. When the offspring is certified as an equilateral, it is taken away from the parents and brought up by a childless couple of the Equilateral group.

Development now proceeds very rapidly. The sons of an Equilateral are Squares, their sons in turn are Pentagons (equilateral), then Hexagons, etc., until the number of sides has become so great that the creature resembles a circle. It is then allowed to call itself Circle and admitted to the

Priest class to which all dignitaries belong. This is headed by the Supreme Circle.

The Equilaterals make up the middle class, composed of shopkeepers, merchants, and clerks. The "gentlemen," officials and scholars, are Squares, Pentagons, and Hexagons. They already rank among the nobility, the Polygons (with many angles), who regard themselves as being well above the common workers' class of the Isosceles.

The extremely underdeveloped ones are constantly at war with each other and fights between these sharp-angled creatures are often bloody. The Circles look on this with considerable satisfaction because it lessens the dangers of overpopulation and dangerous rebellions. Nevertheless, it has happened more than once that the lower classes have revolted against the administration. Since the most dangerous individuals with the sharpest vertical angles are also the ones with the smallest brains, such a rebellion is always led by the less sharply angled Isosceles. This fact was turned to good use when the chief rebel leaders were made to undergo an operation in the hospital increasing their vertical angle — which was already close to 60°— to that very value, so that they could be admitted to the higher ranks. Other leaders were also lured to the hospital where they very treacherously were either made prisoners for life or, if they did not go along with that peacefully, were simply killed. The remaining rebels, deprived of every form of leadership, were egged on to fight each other so that the rebellion ended in a mutual murder party.

The females are line-shaped. It is impossible to determine

from their appearance to what caste they belong and their lineage is therefore controlled with care. An upper-class male will not want to marry a woman from a lower rank for fear that his backsliding will show itself in his progeny. Curiously enough, Polygons seem to take the matter more casually. Convinced as they are of their own excellence, they dare to marry the girl of their choice without paying much attention to pedigrees and certificates.

Generally speaking, the women are unhappy creatures. Their very tiny top lets them have very few brains. Because they are needle-sharp, a collision with them is instantly fatal. Laws have therefore been enacted which they are obliged to follow. Each house has a wide door for men and a little narrow one for women, who are allowed to use only the latter. On the street, women are required to keep up a constant warning cry. Failure to do this means the death penalty. It is further decreed that every woman suffering from St. Vitus's dance, fits, or a chronic cold accompanied by violent sneezing, or any other disease involving involuntary motions, must be destroyed immediately.

In certain states women are forbidden to go out on the streets at all. They must remain indoors all their lives except on certain holidays. But this rule has proved unsatisfactory because continuous confinement makes the victims extremely irritable, thereby causing many more marital fights than usual, of course with fatal consequences.

All this curtailing legislation was enacted on behalf of the women as well as the men. Walking backward, for example, a woman can accidentally spear someone and, unable to with-

draw her barbed posterior from the squirming victim, can also break into pieces herself. A woman's endpoint is particularly dangerous. First of all, she cannot see what is happening on that side, and furthermore her endpoint is dark, increasing the danger of unexpected collisions. The frontal point, which has an organ functioning as both mouth and eye, can always be distinguished easily as the point that sparkles. Many countries have therefore decreed that a woman must keep her endpoint in constant motion. This commendable custom has been given an extra boost by fashion. The ladies from the higher ranks, in particular, are able to make a rhythmic wagging movement with exceptional grace and all the other women try to copy them as much as possible—still, it is always easy to recognize a lady of the aristocracy by her elegance in this respect.

At home the woman presents a constant danger. To make her angry is synonymous with suicide. The women's chamber, entered through the already mentioned women's door, is so narrow that she cannot turn around in it. The husband can enter via a side door, which he can also use to hurry out again if necessary. This type of construction forestalls a great many problems of possibly fatal consequences. Among the lower classes violent quarrels occur frequently, but the dangers are much the same for both sexes because the upper angle of the men is a fair match in lethal power for the sharp extremities of the women.

In the refined societies of Polygons and Circles, it is the custom for the woman to keep her eye and mouth constantly directed toward her lord and master. This adds to the general

safety, because one does not collide as easily with the clearly visible front end of a woman as with her posterior. It does create other problems, however, for having his wife's penetrating eye and especially her constantly chattering mouth aimed at him all the time can drive a man to distraction.

The inhabitants of Flatland never see more than a line, and you may wonder how they manage to recognize anyone! First of all, they use the sense of hearing. The voice of the highly developed Flatlander sounds more aristocratic than that of the common man. But this is not entirely reliable, and besides, some Equilaterals can do a very unfair imitation of a Polygon's voice.

The touch system is better. The introduction of one person to another is done among the ordinary ranks with the customary formula: "May I ask you to feel Mr. X and to be felt by him?" In a shortened version one simply says: "May I feel you, Mr. X?" A rather strange expression, of course, but everyone understands it right away.

"Feeling" means carefully touching one of the angles. As soon as they are of ordinary public-school age, children begin training in this and an adult, after one simple touch, can usually tell the size of an angle pretty accurately.

But recognition can also be accomplished face to face. Since a very thin mist prevails everywhere in Flatland, the more distant objects are always hazy. When one comes face to face with someone else's angle, the sides leading away from it become less clear. This makes it possible, after much practice, to estimate the size of the angle quite accurately.

The feel method is used primarily by the lower ranks of

the population, the view method by the higher ranks. In the exclusive schools of the upper classes, the latter is taught from the very first year.

The life of society in general rests on the regularity of its citizens. Occasionally it happens that a very irregular child appears. If the deviation is not too extreme, some pushing and pulling, done in the clinics, can do much to improve the shape. If the irregularity is too great, the only resort is to kill the misformed creature painlessly. The structure of the entire society would otherwise be disrupted by the existence of such a creature, who might look like a pentagon when seen from one side, while having a much sharper angle on the other. Whether these monsters are really criminals from the moment of birth, or whether they become criminal because of having been ridiculed, rejected, and disdained from childhood on, is an unanswered question: but they constitute a danger for society at large and therefore cannot be tolerated in it.

The state is ruled by the "Circles," who are really polygons with sometimes as many as several hundred sides. It would be difficult to determine the exact size of their angle, even by very careful feeling. The natural law that such a large number of sides denotes distinguished ancestry does not always hold true, because a practice has crept into the highest ranks of the Polygons of having their children, when barely a month old, reshaped in the state clinic. The frame is broken in many places, which increases the many-sidedness. This operation is exceptionally dangerous—only a small percentage of the children survive it. But parental vanity

drives the aristocrats to subject nearly all their children to it. The administration tolerates this since the individual's fertility has been lessened with the increase of his sides and if not enough Circles are born, more must be made.

To us three-dimensional creatures, life in a flat plane seems terribly dull. The entire panorama a Flatlander sees is line-shaped. Also, there is very little color difference, as far as we know. This was different once, as the Square tells us. Chromatistes, a Pentagon who invented the art of painting, launched a color fashion which became extremely popular. All the Flatlanders began to paint their sides in different colors. This not only made life more agreeable, it made mutual recognition easier. There was no longer any need to teach the art of angle viewing. The higher ranks had always excelled in this skill and that superiority over the less-developed creatures suffered a painful setback. This in turn led to heightened jealousy and friction among the classes, together with an increasing amount of deceit. For example, a simple Isosceles would sometimes paint himself with many colors, making others see him as a respectable Pentagon. As was predictable, the bubble finally burst. The Circles and Polygons, who had their doubts about the outcome of an open battle with the sharp-angled ones, resorted to an underhanded plot. Under the guise of introducing a very democratic color law, they held a council meeting in which the Head Circle, named Pantocyclus, addressed the crowd. By emphasizing the disadvantages of the new law he overruled the hesitant middle class. The Isosceles who were already close to being Equilaterals were no longer particularly inter-

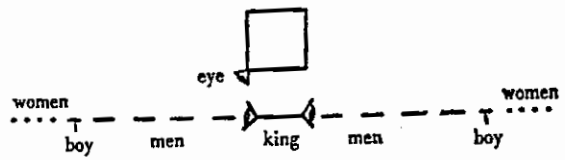
ested in equalization of all ranks and declared themselves to be against democratization. When things had reached this stage, a prearranged signal was given and a terrible bloodbath took place. The ruling parties ordered a charge on the Isosceles by an army of very sharp-angled soldiers, who knew little or nothing about the issue, and by a special regiment of orphan girls. No second charge was needed, because the confusion in the ranks of the attacked was such that they thrust around blindly, inflicting countless casualties on themselves. The fact that the very sharp-angled soldiers fighting for the aristocrats had also decimated their own ranks in the confusion of the battle was not unwelcome to the Circles.

Since that time, painting has been abolished and the ban strictly enforced. Peace has returned and the world of Flatland is ruled strategically as of old by the Circle group.

2 *Dream Vision of Lineland*

On the next-to-the-last evening of the year 2000 of the Flatland era, the Square, who has been explaining the particularities of Flatland to us, had a vision. He saw a line-shaped land, in other words, a one-dimensional world.

The Square visits Lineland.



The creatures living on this endlessly long line were little lines. They moved back and forth along it and of course were not able to pass each other.

Something like a rush or a roar would go out from the entire line, changing occasionally into a chirping or a warbling, then suddenly stopping again as everything became still once more.

One line was longer than the others, and naturally thinking this was a woman, he addressed her as such and asked what the excitement was all about.

The individual he had addressed said crisply: "I am not a woman. I am the ruler of this world, the king of Lineland."

This potentate could of course not conceive of anything existing outside his own line-shaped world. As far as he knew, his line made up the entire existing space. He could not imagine where this stranger, suddenly looming in front of him, had come from since he, the king, could have no concept of a direction perpendicular to his own world.

The little lines in his world are the men, the dots the women. The visitor saw four men standing on either side of the king, followed by a shorter little line, a boy, in turn followed by four women. They could all move back and forth over a very short piece of their world, and—their two eyes being located with their mouths at each extremity—no one ever saw anything but a dot.

All this seemed terribly dreary to our Square and he wondered whether any sort of family life was possible here. With this in mind he asked the king a somewhat personal question about the state of his family's health.

"Oh," the king replied, "my wives and the children are all hale and hearty."

This surprised the Square very much because only men were standing on either side of the king. He said that he did not understand how His Majesty could ever see his wives or approach them, and that would seem to be quite essential for a marriage and children.

The king found that a little silly. "The birth of children," he explained, "is too important to depend on something as haphazard as contact"; and he went on to tell how each man has two mouths or voices, a bass at one end and a tenor at the other.

"I wouldn't bother telling you this," he said, "if I hadn't noted in the course of our conversation that I did not hear your tenor voice."

The Square explained that he had only one voice, which led the king to observe: "That confirms my impression that you are not a man, but a female monstrosity with a bass voice."

The women in Lineland only have one voice, you see, and it is either a soprano or a contralto. A marriage materializes through the harmonizing of a male's bass and tenor voices with the soprano and contralto of two females, which in turn results in the always simultaneous births of two girls and a boy. By means of this arrangement on the part of nature, the balance of the sexes is always preserved in Lineland.

Next, the Square asked the king how he was able to recognize his subjects, and the latter immediately proceeded to show how he could call his wives, who at that time were each

596 miles 397 yards 33 inches away from him — one to the north, the other to the south.

The king put both his voices into action. On hearing them the women took note of the short time difference between the voices, which indicated to them, after a brief calculation based on the velocity of sound, that the distance between the king's two mouths was 6.083 inches. As the king explained, the calculation was not really necessary for his wives because they had already known their husband for a long time, but in this way one can estimate a fellow Linelander's length quite accurately in the course of conversation.

Then the Square wanted to explain to the king how very limited his life in the line-shaped world was. He pointed out how a man with his type of vision could never distinguish anything more than a dot while he, a Flatlander, could see the difference between a dot and a line. Of course the king was unable to understand this. "How can you claim," he asked, "to see a line, in other words, the inside of a man?"

The Square said that he could see four men, a boy, and then four women on either side of the king, but this did not impress His Majesty in the least, because even the smallest boy in his country would know that.

The Square then took a different tack and explained how, in addition to the north-south one, there was still another direction. Naturally, he could not satisfy the king's request to show it to him. To see it, the king would have to step outside his line.

"Step outside my line?" the king cried out. "Outside my world? Outside of space?"

"Yes," the Square answered, "outside your world, outside of your space, because your space is not the real space. The real one is a plane and yours is only a line."

It goes without saying that this went beyond the king's comprehension altogether. A further suggestion by the Square that the king should try, just once, to move in the direction of his side rather than that of his extremities, did nothing to clarify the matter.

"How can anyone move in the direction of his own inside?" the king asked. Now the Square was forced to resort to action. He pushed himself in front of and past the king all the way through Lineland with the result that the king kept seeing him reappear and disappear. But the king regarded it as little more than a magic trick beyond his comprehension.

When the Square gave him a sideways shove in his inside, the king did experience a painful sensation in his stomach, but this still failed to provide him with insight into the existence of a direction perpendicular to his line.

Then the Square got angry and started calling the king stupid and simple-minded, and when he again pushed his way through the king's world, and the latter, now thoroughly aroused and furious, wanted to ram the nasty intruder, the Square awakened in time from his strange dream.

3 *The Visit of the Sphere*

The following day was the last day of the year 2000. On the stroke of twelve the year 2001 would dawn and with it

the third millennium of the Flatlander's era. The Square, who is known to have been a great mathematician, often taught his grandson geometry. He really enjoyed doing this, for the boy, who had been living with him since the time he lost his parents when still very young, was a purely constructed Hexagon with a clear intellect and an exceptional talent for mathematics. And that is how the Square happened, late in the evening of this last day of the millennium, to be teaching the boy about the connection between arithmetic and geometry. As teaching material a number of squares with sides of 1 inch each had been placed on the ground. He now arranged nine of them into a larger square, each side of which was 3 inches long, and pointed out that it was very easy to calculate the number of square inches of the square, even though the inside of a square can never be seen. "In fact," said the Square, "the inside amounts to 3^2 or 9 square inches."

The small Hexagon thought about this for a little while and then said: "But Grandfather, you have also taught me to raise numbers to the third power. Then 3^3 probably has a geometric meaning too."

The grandfather explained how that could not be. "Whenever a point is moved over a distance 3, that point describes a line of length 3. If now the line is moved in a direction perpendicular to itself over a distance 3, a square results which can be represented by 3^2."

"Fine," the grandson said. "If the square, which is 3^2, is moved over a distance 3 in a direction which I do not see, something has to come into being which can be represented by 3^3."

"But that is nothing, after all!" the Square said.

"I can't see it either," the boy answered, "but I am merely continuing my calculations in the same way."

"In geometry we must always stick to what is logically possible," the grandfather stated and sent tbe boy to bed.

"The boy talks nonsense," the Square then said, more to himself than to his wife, "a stupid boy."

"He is not stupid," his wife bounced back. "The boy has a bright mind. And you'd do well to remember that, according to the law, you as a Square owe respect to a Hexagon, even if he is your own grandson."

"What he said is still nonsense," the Square muttered again.

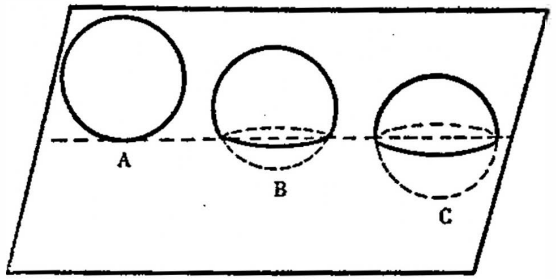
"No, it isn't! The boy is certainly not stupid. What he said wasn't nonsense at all!"

Who said that? Where did that strange voice come from? There was no one to be seen, but suddenly a dot materialized which changed into a small circle. That little circle grew bigger and bigger until it had a center line of about 12 inches.

For a moment the Square and his wife were speechless. It is strange enough to be visited at home by a Circle, but when such a dignitary makes his entrance in such an unaccountable way — that is extremely odd.

"I should like to talk with your husband alone. I have a message for him," the visitor stated.

The wife, who, like all wives in Flatland, was very humble and obedient, took her leave and went to her room. It was just twelve o'clock; the new millennium had dawned.



The Sphere visits Flatland.

"I am not a Circle," the stranger explained, "or rather, I am a more complete Circle than you can visualize in Flatland. I am many Circles in one."

"Where do you come from?" the Square wanted to know.

"From space," was the answer.

"But I just saw you enter into space," the Square returned.

"Well, what you call space. But what do you, a Flatlander, really know about space? You only know two dimensions, length and width, but the space where I live has three: length, width, and height."

"Oh," the Square said, "we also speak of length and height, but that is the same to us as length and width. There are only two dimensions, after all."

"Not with us," the stranger said again. "I came out of a third direction which you cannot see, a dimension perpendicular to your two dimensions. I am the most perfect creature imaginable, an Over-Circle you might call me—Sphere is what we call it in Spaceland."

"You mean to tell me that a direction exists which is not north-south, not east-west, but perpendicular to that? Perpendicular to both directions at the same time?" the Square cried out. "Then please show me that direction."

"You cannot see it," the Sphere said. "To do that you would have to have an eye in your inside."

"An eye in my inside? In my stomach? But I wouldn't be able to see with that."

"I can look into your insides," the Sphere resumed. "Your inside is completely open to me. When I descended here I could see your entire house. I saw the occupants in their rooms; I could look into your closed closets." After a short pause he went on: "I have been observing you, dear Flatlander, for several days now. I saw a dream vision of Lineland in your brain. You were talking with the king of that country and you tried to convince him that you were able to look inside him. In the same way I, who am living outside of your flat plane, can look into your insides, because it is open to my view just as the inside of a Linelander was for you."

To make this clearer, the Sphere disappeared from the plane of Flatland and shortly thereafter gave a little shove in the Square's insides. He also took some objects out of the closed closets and deposited them in front of his host.

"Now I will descend once more into your space," the Sphere said. Again the Square saw a dot appear which turned into a small circle and slowly grew in size.

"I can see nothing of your descent," the Square said. "I can only see a Circle which keeps growing."

"You cannot see my ascent or descent," the Sphere

explained. "It occurs in a direction which is invisible to you. If one of your Circles were to visit Lineland, what would the king of Lineland see?"

"Well," the Square said, "first of all he would see a dot, which would change into a line. At least he would be able to determine that by means of the sound. The line would grow until it had reached its maximal value and then would diminish again."

"Exactly," the Sphere said. "The Linelander always sees the next cross-section of the Circle, and that is also how it is with you. You always see the succeeding cross-section of me and these cross-sections are circles."

"In other words," the Square resumed, "I can only understand this by means of an analogy?"

"Precisely," continued the Sphere. "By means of an analogy. And since we are talking of analogies, just suppose that a dot is moving northward, leaving a brightly lit trail behind it. What would you call that trail?"

"A straight line."

"And how many ends does a straight line have?"

"Two."

"Now just imagine," the Sphere continued, "that the straight line moves parallel to itself over an equally great distance to the east. What name would you give the figure which materializes?"

"A square."

"And how many sides does a square have? And how many vertices?"

"Four sides and four vertices."

"Fine. Now put your imagination to work," the Sphere continued, "and imagine that the square is moving parallel to itself in a direction which is inconceivable to you, entirely outside of Flatland; something then develops which we call a solid body. You cannot visualize it but you can calculate analogically how many vertices it has. Just remember: a single point consists of one dot. A line has two end-points, a square has four vertices. The series which results, 1, 2, 4, is apparently a geometric progression, and the next term therefore is . . .?"

"Eight," the Square said.

"Right. You reasoned correctly. We call the body which materializes from the sideways movement of the square a *cube*. And a cube has eight vertices."

"And how many sides does the cube have?" the Square asked.

"That too," the Sphere replied, "we can discover by means of analogy. The border of something is always one dimension behind the thing itself. Accordingly, a line is bordered by dots, a plane figure by lines, and a solid body by . . ."

"Planes," the Square supplied.

"Right," the Sphere granted. "A point has no dimensions and no borders either. Thus, number of borders: zero. A line is bordered by two dots, a square by four lines. It is an arithmetic progression: 0, 2, 4, and therefore . . ."

"And therefore," the Square took over, "a cube is bordered by six squares."

"So you see you discovered the answer yourself, through reasoning," the Sphere summed up.

4 *To the Land of Three Dimensions*

Afterward, the Square could not have said how long the Sphere's visit lasted. Did it really happen, did the Sphere really lift him forcibly out of his plane, his Flatland, and take him to Sphereland, where he could view his world from the outside? He saw his own house as he had never seen it before. He saw the various rooms next to each other, his four pentagonal sons, both his hexagonal grandsons, his daughter, his wife, the servants. Outside, two policemen were strolling by. He saw the entire street, the theater, and in the distance a large polygonal building, the Council Room of the states of Flatland.

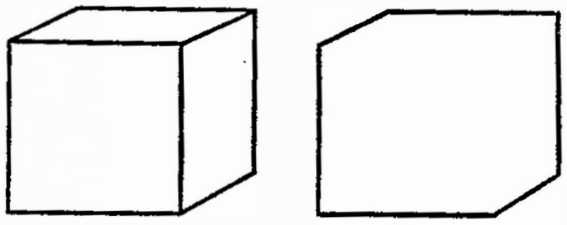
He came closer. A session was in progress. As was customary at a turn of the century, the states had gathered to protest against demonstrations from Spaceland. It was a matter of public record that at the beginning of the year 1 and also of the year 1001, a creature from beyond Flatland had paid a visit to the States Conference to argue for the existence of a third dimension. This time, too, a visit was expected. And it took place. The Sphere descended into the packed Council Chamber to carry out his assignment. Once every thousand years a delegate was sent out to try and convince the Flatlanders of the limitation of their world and of the existence of a three-dimensional world, and this time it was the Sphere's turn to undertake the mission.

His sudden appearance in the Council Chamber caused much confusion. The Supreme Circle ordered the guard to attack the stranger, but the latter escaped into an invisible dimension.

Thereupon the Council of Wise Circles resolved that no one was to hear about these amazing developments. Since the Circles were the only ones considered capable of keeping secrets, it was decided to kill all others who had been present. Only a few Chamber attendants were actually involved, and as a precautionary measure those selected for that post were less valuable Isosceles with exceptionally sharp vertices. The alerted guard arrested the unfortunates and carried out the verdict. There was only one other person present, the recording clerk—a Square who was recognized with great shock by our Square as his own brother. The poor man, who could only be accused of having had to be present at this session because of his official function, was not sentenced to death but only to solitary confinement for life.

After completing his trip, the Sphere returned to his guest. He showed him a cube, but since the Square was not used to seeing perspective he mistook this body for an "Irregular." Still, on the basis of what he could see and by applying the analogy, the Square succeeded in getting a pretty good image of the world of three dimensions. Finally he

A Cube (A). *The Square mistook it for an Irregular* (B).



A B

turned to the Sphere and said: "From the land of two dimensions one can see the inside of Lineland's inhabitants. You have taken me with you to the land of three dimensions, from which I can see the insides of my compatriots. Now I should like to go to the land of four dimensions, from which I can take a look at your insides."

This was too much for the Sphere. He considered himself the most perfect creature imaginable, and now this puny, two-dimensional creature wanted to look all the way into *him.* Besides, in his opinion a land of four dimensions was impossible. When the Square told him that the analogy could be continued and that a four-dimensional body could therefore be predicted, an Over-Cube which according to the progression: 1, 2, 4, 8, 16 must have sixteen vertices and according to the progression: 0, 2, 4, 6, 8, eight side-cubes, and when he asked whether no fourth-dimensional visitor had ever come down into the land of three dimensions—then it all became too much for the Sphere. He did admit there were rumors that some people had visits from a very strange creature out of another world, but no one had ever taken this seriously. It was generally ascribed to hallucinations arising in the brains of sick individuals, the result of their "confused angularity"!

When the Square persisted, the Sphere rudely shoved him back in his study, where he gradually returned to his senses from the daze brought on by shock—not certain whether he really made the trip to the land of three dimensions or only dreamed it.

He went to his bedroom and fell into a deep sleep. And

here he dreamed again. The Sphere took him to the country of zero dimensions, consisting of only one single point who was humming along self-sufficiently and happily, thinking he was all there was in the world—and actually he *was* everything in it, because his world did not have one single dimension.

5 *Dishonored*

The following day, New Year's Day, our Square found himself facing the new millennium with joy and anticipation. This was to be the millennium of enlightenment. There would be epoch-making new ideas, starting with a better insight into the existence of a world of three dimensions. He felt himself called to preach the new doctrine. The concept "Upward, not northward," engraved in his mind, would be his guideline.

With whom should he start? With his wife? At the very moment he was considering this, he heard the voice of a town crier on the street announcing the Council's decision that anyone trying to poison the mind of the public with claims of getting reports from another world would be imprisoned or killed.

This was not to be taken lightly, but the Square felt self-confident and strong. He would not make any loose statements, but he could give a scientific demonstration, an argumentation, and that would change things considerably.

Nevertheless, it might be wiser not to start with his wife,

much less with one of his four sons. Not one of them had more than an average aptitude for mathematics, and besides, he was not sure that their filial love would outweigh their sense of obligation to report their father to the prefect—their father being a mere Square and they themselves Pentagons!

He thought it best therefore first to test his grandson, who had undeniably great mathematical aptitude and had made such intelligent comments. He had the boy come to him and tried to explain that the third power of three could indeed have a geometric meaning. He told how a square can originate from a line, and how, by moving this square upward and not northward . . . But already he found his words beginning to lack conviction, for when he wanted to make himself clear by moving a square, he found himself pushing the square in an arbitrary direction and repeating, "Upward and not northward"; but of course he could not move the thing upward.

At that moment the town crier came again within earshot, threatening all who propagated revelations from another world with the most terrible punishments. The boy also heard this and understood it all too well. He became terrified, and bursting into tears he said that he had not meant to imply anything and that something like this could have no geometric significance at all.

Any further attempts to convince the boy of the correctness of the concept he had suggested earlier were of no avail. The boy was afraid that his grandfather had it in for him and, terribly frightened, he ran out the door.

The Square thought it best to keep his view to himself, but it was not easy, because anyone who has acquired such an

insight naturally burns with the desire to make it known to others. And so it happened that at a gathering of the Natural Sciences Association—and, worse, at the very home of the prefect where one of the members was giving a lecture on Providence's foresight in limiting the number of possible existing dimensions to two, thereby leaving it up to Providence alone to view the innermost parts of things—it was at this gathering that he raised his voice and in a fiery dissertation argued the existence of the third dimension. He told about the Sphere's visit and about his own trip to Spaceland.

Of course he was arrested, and inasmuch as he could not meet the only requirement asked of him, to point out the direction of "upward but not northward," he was sentenced to life imprisonment.

In solitary confinement in a dungeon he was allowed to be visited once a week by his brother, who, as we know, had also been sentenced there for life. With him the Square spoke regularly of Spaceland, but without any success. His brother, who had actually been present during the Sphere's visit to the Council Chamber, did not want to accept his theory.

Lacking any followers, the Square began to write his memoirs in the hope that these might at a later day come into the hands of more enlightened minds who would be able to see the truth of the existence of a space of three dimensions.

And these are the memoirs which under the title of *Flatland* have been published in an enlightened world, now fully capable of understanding their meaning and significance.

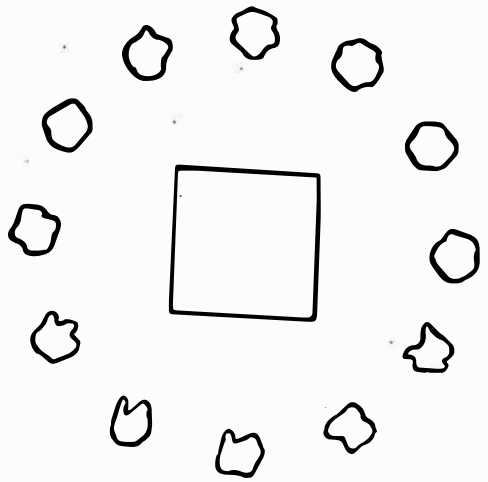
SPHERELAND *A Fantasy About Curved Spaces and an Expanding Universe*

by A HEXAGON

PART I The Straight World

1 Changed Times

More than seventy years have passed since my grandfather, the famous Square, published his ideas about other worlds. I consider it my duty to show how greatly these ideas have now changed. And I feel myself called upon to do this because a slight but constant feeling of self-reproach has been bothering me and I have not been able to reason it away completely. Wasn't it I who, against my own better judgment, deserted my grandfather? At the time of the discussion in his room when he tried to explain to me the possibility of the existence of a third dimension, I put him off and acted as if I thought he was talking nonsense—and all the time I knew perfectly well that he was right. Even though I try to soften this by telling myself that I was only a child and therefore not responsible for my actions, and that I was afraid of the consequences, self-reproach has stayed with me through the years.



My grandfather's statue in a circle of twelve trees.

My grandfather's honor has now been restored. Unfortunately he did not live to see this, but his descendants have done everything in their power to right the wrong that was done. A bigger-than-life-sized monument has been set up for him on the market square of the town where he was born. It is surrounded by a circle of twelve trees and this circle represents the highest wisdom.

Each year a delegation including several professors of mathematics and physics, the mayor of the town, and notable figures in the field of science and politics pays an official visit to the monument. A speech is made recalling how backward society was when it believed it could restrict or curtail science.

But great changes have occurred in other areas too. The

backwardness of women has disappeared almost entirely. The notion that a female is a stupid creature because of her small brain area has turned out to be false. Some women now even study and get degrees from a university. Man no longer has a monopoly over science, even though nature did create woman first of all for marriage. It is she who by virtue of her great gifts of love and devotion has been destined to raise children and to dedicate herself to homemaking. This takes up such a large part of her mental life there is usually no room left in her intellect for study of science. The really important inventions and discoveries will undoubtedly continue to be made by men!

Even woman's temper tantrums, once so greatly feared because they could cause such disasters, are mostly a thing of the past. Nature has endowed woman with the gift of self-control. This was not understood formerly and girls' education helped make them think they were totally lacking in it. Therefore, no feeling of responsibility was developed and a woman remained convinced that she could indulge in rages with impunity. After all, she was only a woman! Now she knows, however, that like the male offender she will have to answer for her behavior before a judge.

Another very important factor enters here. As a rule, the judge will order an examination of a female defendant's mental capacity in order to determine whether she is responsible for her actions. With us such an examination simply consists of checking the external shape of the defendant, and a woman is mortally afraid of even the smallest deviation being found in her "line." It would be a blow to her pride.

She much prefers to be held fully responsible, even though she has to bear the consequences.

The danger that woman presented to society largely disappeared with the innovation of the shoe for women. In public, every woman wears a little shoe on her posterior. You can even collide with it at top speed without seriously hurting yourself. The law that a woman must constantly move her endpoint back and forth has been repealed, there no longer being any need for it, and a woman no longer needs to sound constant warning cries when out in public. The traditional way of moving through the streets did remain in vogue for a long time among women of the upper classes. They were proud of the graceful, rhythmic movements of that part of their anatomy, after all! But now even this has disappeared. Aristocratic ladies became such objects of ridicule for boys in the street that they began to feel silly; moreover, it happened more than once that their rhythmic movements made them lose their shoe. Then the young urchins would make fun of them, yelling "Cinderella, Cinderella!"

To understand this, you must know that we have an old fairy tale about "Cinderella." She was a beautiful young girl, very straight and slim of line. And she had two stepsisters who were also very beautiful, but not as beautiful as Cinderella. They were, in fact, considerably heavier of line. Naturally this made them very jealous and so they forced her to do all the dirty work. She had to fix the meals, scrub the house, and clean out the fireplace. And because she was always brushing and blowing in the ashes, she could never get to look like a lady.

As fate would have it, an official court ball was to be given in honor of the crown prince, who had just reached marriageable age. The most beautiful girl in the world was to be chosen for him, and all young ladies who thought they might qualify for princess were invited to come to the palace at nine o'clock. All young men between eighteen and twenty-two were also invited. Their particular qualification was that they should have at least six sides, but control over this was not very strict.

Cinderella would have liked very much to go to the ball, but her wicked sisters laughed at her and said that she could not go without a dancing shoe. It was the custom that ladies at a ball were to carry a shoe at all times, since serious accidents might otherwise occur with the rhythmic back-and-forth movements in dancing. And so Cinderella stayed at home. She sat by the fireplace and dreamed of the prince, a handsome young twenty-four-sided polygon who had been the topic of everyone's conversation of late.

While she sat there brooding, a neighbor came in and said: "Dear child, I will give you a dancing shoe such as no one else possesses—lovely, elegant, and transparent. But first you will have to fix yourself up." When Cinderella had washed herself thoroughly she put the little shoe on. It fitted perfectly. She thanked the kind woman and hurried off to the palace. The neighbor called out after her that she would have to take care to be home before midnight, because the little dancing shoe was made of some chemical material which remained hard and solid for only three hours, after which it would start turning into a jellylike substance. And she would

look ridiculous with such a gooey mess stuck on her body. Actually this was not true, because the shoe was made of beautiful glass that could withstand centuries. But her benefactress was shrewd. She knew a good deal about people and understood that the prince, who would of course fall in love with Cinderella, would receive a shock when his lovely dance-nymph suddenly disappeared.

And that is exactly what happened. When Cinderella made her entrance, the festivities were already in full swing. Everyone looked up and the prince immediately hurried over to her. In fact, she made a spectacular impression on all the young men present. Her glass slipper set off her lovely body line very nicely. The prince danced only with her until the clocks struck twelve. Cinderella started up, tore herself free, and ran off. The palace doors were closed but she escaped through the ventilator. The prince, who had run after her, was much impressed by the fact that she could go through such a narrow opening. Of course he could not follow her, but what was that? He had seen the small, delicate shoe which she had lost during her escape through the narrow hole. He took it with him and swore that he would marry the owner of that shoe and no one else.

The next day he went through town, preceded by heralds who informed the people of the prince's intentions. Every young lady could report in person to try on the glass shoe. Many had to admit with disappointment that the shoe was amazingly delicate and small. Cinderella's sisters tried it on, but of course without success. Then Cinderella asked if she might try it. Her sisters laughed at her and said scornfully:

"You, Cinderella? You mean that you want to marry a prince?" They hadn't recognized her the night before, but they had jealously seen how the prince danced constantly with the same beautiful young lady and would not pay any attention to them. That this young lady could have been Cinderella did not occur to them.

But Cinderella was also allowed to try the shoe on, and to everyone's total surprise the little shoe fitted as if it had been made to order. And now the prince recognized his beloved and brought her triumphantly to the palace. They were married a short time later and lived long and happily, with many children—nice twenty-five-sided polygons and lovely, slim girls.