<u>GENERAL RELATIVITY:</u> The Theory of Gravity

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Appendix D: Gravitation and the Principle of Equivalence
Appendix E: The Twin Paradox

APPENDIX D

GRAVITATION AND THE PRINCIPLE OF EQUIVALENCE

EINSTEIN'S THEORY OF SPECIAL RELATIVITY is very powerful in scope, because it has things to say about the foundations on which physics is built. All of physics must be carefully scrutinized to see if it conforms to relativistic requirements. Thus the electromagnetism of Maxwell was found to pass the test, and was actually partially justified and strengthened by the relativistic point of view. On the other hand, classical mechanics had to be revised in order to satisfy Einstein's demand that the fundamental laws be invariant under the Lorentz Transformation. Here we examine Newton's law of gravitational attraction, summarized by the central force

$$F = \frac{Gmm'}{r^2} \qquad \text{(D-1)}$$

which varies as the product of the two masses involved and inversely as the square of the distance between them. We would like to know if this law passes the test of relativity, or if instead a new theory of gravity is required.

The questioning of Newtonian gravitation at first seems unnecessary or even arrogant, since calculations made on planetary motion using Newton's formula agree with experiment to a very high degree of accuracy. In fact two planets were discovered by the deviations they caused in the calculated orbits of other planets. Yet from a relativistic point of view, Newton's law of gravitation cannot be correct as it stands, but needs at least a partial modification!

First of all, how should one interpret the distance r between two masses? Who is supposed to measure this distance? Since lengths have lost their absolute character in relativity, an observer on one mass may measure a different distance than an observer on the other mass. We have no reasonable way to specify what distance should be used! It is also implied in the formula that if you want to know the force between two objects now, you should put in the "distance between them" now. But how does one mass know where the other mass is now? If no signal can travel with infinite speed, a given mass can only feel where another was at an earlier time – at least the time it takes light to travel between them.

Another interesting comment one could make about Newton's formula is that the force depends on the particle masses. Yet in relativity theory mass is just one form of energy. Why should gravity act only on that part of the energy contained in mass? Might it not also act upon (and be caused by) kinetic energy and massless particles like photons?

Finally, there is a fact about gravitation which doesn't disagree with Newton's theory, but which isn't explained by it either. That is the fact that in writing the equation of motion of a particle in a gravitational field, the mass of the particle cancels out, and so does not influence the motion. This of course is just "Galileo's experiment" of dropping two different masses and noticing that they accelerate equally. The mass m in the force law $\frac{G\,m\,m'}{r^2}$ we could call the gravitational mass, whereas the mass m in the "inertial-force" $m\,\vec{a}$ could be called inertial mass. The fact that these masses are equal, or at least proportional through the constant G, is not explained in Newton's theory of gravity.

For these and other reasons, after completing special relativity Einstein went to work on a new theory of gravitation. This led to his general theory of relativity, published in 1915. The theory overcomes the objections to Newton's theory, and also has the required property that it reduce to special relativity in the absence of masses, and that it reduce to Newton's gravitation for non-relativistic objects in weak fields. Einstein's special relativity was the result of carefully developing the idea that light moves at the same velocity in all frames. An equally simple but powerful idea, which he called the "Principle of Equivalence," guided him in the construction of general relativity. By using this principle, along with some beautiful mathematics describing curved space as invented by Gauss, Riemann, and others, Einstein proposed a theory of gravity which accounts for gravitational phenomena in terms of a curved "non-Euclidean" geometry of space-time. Although

a retracing of the general theory is far beyond the scope of this book, the equivalence principle by itself leads to some interesting results. We discuss it here for its own interest and because it is useful in understanding the twin paradox, as taken up in Appendix E.

There are several ways to state the principle of equivalence, some of which are not equivalent to others! The reader is referred to the references for all but the single approach we take here. Imagine two spaceships, one of which is uniformly accelerating in empty space without any gravitational field, and the other standing at rest in a uniform gravitational field, as shown in Figure D.1. The principle of equivalence then claims that an experiment performed inside the accelerating ship will give the same result as an exactly similar experiment inside the ship at rest in the uniform field.

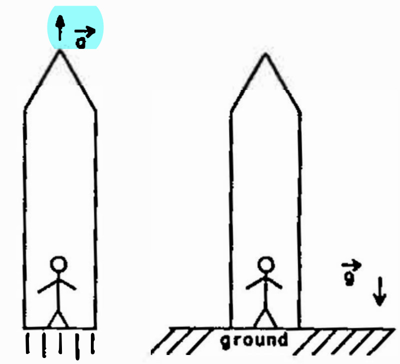

Figure D.1

A key word here is "inside," since you could clearly distinguish between the two situations by looking outside to see if you are standing on some large mass or not. The principle is a way of expressing the observed fact that inertial and gravitational mass are equal. In the accelerating ship, it is an inside observer's inertial mass which "causes" him to press against the floor; in the stationary ship it is his gravitational mass which performs this function. The equivalence of these two situations is quite reasonable for mechanics: intuition and exact analysis agree that the motion of an object inside is the same in either case. What is not so clear is that the principle applies to experiments with electricity, light, atomic and nuclear physics as well as mechanics. Yet Einstein decided to pursue this principle, supposing it to be universally valid, to see where it would lead him. We will use the principle here to deduce two effects of gravity which aren't contained in Newton's theory: the effect of gravitational potential on the rate of clocks, and the bending of light in a uniform field. These effects are deduced by considering two experiments with light waves.

The first application of the equivalence principle to the behavior of light waves is the derivation of the so-called "gravitational red-shift." This phenomenon can be understood as an influence of gravitation on the rate of clocks, quite apart from the special relativity effects met with earlier. To derive this effect, we make use of the device of deducing the result of an experiment performed in a uniformly accelerating rocket, and then claim by virtue of the equivalence principle that the experiment would give the same result in a rocket at rest in a uniform gravitational field. The "experiment" is simple in principle, but would be difficult to carry out in practice.

At the top of the accelerating rocket is an observer who shines a flashlight at another observer at the bottom of the rocket, as shown in Figure D.2. For simplicity we assume the flashlight emits monochromatic light, and also that the distance traveled by the bottom observer while the light comes to him is small compared to the length of the ship. It follows that the time it takes for the light to reach him is about $t = \ell/c$, where ℓ is the

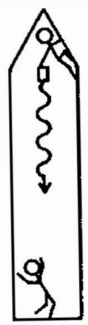

Figure D.2

distance between the two observers. But during this time the bottom observer has attained a velocity $v = at = a\ell/c$ with respect to the velocity of the flashlight when the light was emitted. He is moving toward the source, so he will observe a blue-shift due to the Doppler effect. Since we have already assumed his velocity is small, so as to neglect the distance he travels compared to the distance ℓ, we can use the non-relativistic Doppler formula to give

$$\nu_{\text{observed}} = \nu_{\text{emitted}} (1 + v/c) = \nu_{\text{emitted}} (1 + a\ell/c^2). \qquad \text{(D-2)}$$

Clearly if instead the bottom observer were to shine a monochromatic flashlight at the top observer, the top observer would observe a red-shift, since by the time he received the light he would be moving away from where the source was when the light was emitted.

Now by means of the equivalence principle we can claim that the same effects would be observed in a rocket at rest in a uniform gravitational field, if we substitute the acceleration of gravity g for the rocket acceleration a. That is, if the observer at the top of the ship were to shine a frequency ν_{emitted} toward the lower observer, the lower observer would see a frequency $\nu_{\text{observed}} = \nu_{\text{emitted}} (1 + g\ell/c^2)$, a blue-shift, whereas the top observer would see a red-shift if he looked at a light beam sent off by the lower observer. But in this case we can hardly blame the shift on Doppler, because neither observer is moving. We have to invoke some other explanation to understand the change in frequency in a rocket at rest in a uniform field. This situation is reminiscent of the muon decay problem of Chapters IV and V, where it was necessary to explain why the muons failed to decay before they had penetrated the atmosphere. The fact of penetration held in both the earth and muon frames, but the reason for the fact was different in the two frames. Observers on the earth explained it by saying that the muon clocks ran slow, whereas the muon explained it by saying that the atmosphere was thin because of the Lorentz contraction.

How then can we explain the blue-shift seen by the man at the bottom of the stationary rocket? If we think of the atoms which radiate the light as clocks whose rate is indicated by the frequency of their emitted light, the observer at the bottom would be forced to conclude that these clocks are running fast compared to similar clocks beside him. His own clocks radiate a certain frequency, while atoms higher up radiate a higher frequency. The observer at the top would agree with his judgment. He sees a red-shift when he looks at atoms below him, so he would say that his own clocks are running faster than those below him.

If atomic clocks up high run faster, it is of course true that all clocks up high run faster. For suppose a clock at the top of the rocket has a luminous second-hand emitting light of frequency $\nu = 5 \times 10^{14}\ \sec^{-1}$, corresponding to a yellow color. Then in one complete revolution of the second-hand $60 \times 5 \times 10^{14} = 3 \times 10^{16}$ waves will be emitted. The observer at the bottom must also see 3×10^{16} waves/ revolution, since none are created or destroyed in transmission. But the frequency of the waves would increase by the factor $(1+g\ell/c^2)$, so it follows that the second-hand of the clock at the top appears to complete a revolution in less than 60 seconds to the man at the bottom, by the exact same factor.

The gravitational effect on clock rates is not so paradoxical as the time-dilation effect for moving clocks, since both the upper and lower observers agree that the upper clocks run faster than the lower clocks. It is interesting that the quantity $g\ell$ is just the difference in gravitational potential $\Delta\phi$, in a uniform field. In terms of $\Delta\phi$, the potential energy $(mg\ell)$ of a particle having mass m is $m\Delta\phi$. The gravitational effect on clocks can therefore be written

$$\nu_{observed} = \nu_{emitted}\,(1 + \Delta\phi/c^2) \qquad \text{(D-3)}$$

where the sign of $\Delta\phi$ is chosen to make light from higher clocks have a higher frequency.

It is easy to calculate the magnitude of this effect on the earth's surface. Using a uniform g of about $9.8\ m/sec^2$, a clock on top of 29,028-foot Mt. Everest would run faster than a similar clock at sea level by about one part in 10^{12}, which amounts to a gain of one second in 30,000 years. Within the past few years it has become feasible to perform experiments of this accuracy, using the Mössbauer effect, which allows an extremely sensitive determination of frequency changes in γ-ray photons. This effect was used by Pound and Rebka* in 1960 to measure the gravitational effect on the frequency of 14.4 kilovolt photons emitted by the isotope Fe^{57}. They used a tower 74 feet high, for which we would calculate a shift of $\Delta\nu/\nu = g\ell/c^2 = 2.5 \times 10^{-15}$. They observed a shift, and found that

* Pound and Rebka, Phys. Rev. Letters 4, 337, 1960

$$\frac{\Delta\nu \text{ experiment}}{\Delta\nu \text{ theory}} = (1.05 \pm .10),$$

which is excellent agreement, especially considering that the experiment was so delicate that a 1° temperature difference between the top and bottom of the tower would have destroyed the effect, due to Doppler shifts caused by different velocities of the nuclei in the two positions.

If we assume that the factor $1 + \Delta\phi/c^2$ is also valid for non-uniform fields, for which $\Delta\phi \neq g\ell$, the frequency shift of light from the sun and stars can be used to test the theory. The change in potential energy in going from the surface of the sun to the surface of the earth is

$$\Delta\phi = -\frac{G\,M_{sun}}{R_{sun}} + \frac{G\,M_{earth}}{R_{earth}} .$$

The second term is much smaller than the first, so the frequency shift is:

$$\frac{\Delta\nu}{\nu_{emitted}} = \frac{\nu_{observed} - \nu_{emitted}}{\nu_{emitted}} \cong -\frac{G\,M_{sun}}{R_{sun}c^2} \cong -2 \times 10^{-6} ,$$

or a red-shift of two parts per million. White dwarf stars, with nearly the sun's mass compressed to the size of the earth, have a red-shift about 100 times as large. Shifts have been observed from both the sun and white dwarfs, and are in agreement with the theory, although there is considerable error due to other causes of line-shifting. The shifts are found, of course, by comparing observed atomic spectra with spectra from laboratory sources.

A second application of the equivalence principle is the calculation of the bending of light in a gravitational field. In this experiment, a flashlight is aimed sideways in each rocket, as shown in Figure D.3. We can analyze the motion of the light beam in the accelerating ship, and then claim the results must be the same in the rocket at rest in a uniform field. Making again the assumption that the acceleration is small enough so the rocket doesn't attain relativistic velocities while the light travels from left to right, we know that if the light takes time t to cross the ship, the ship will have moved a distance $\frac{1}{2}\,a\,t^2$.

Therefore, although the light beam moves in a straight line as seen by inertial observers, it moves in a curved parabolic path as seen by observers on the accelerating ship. If x and y are the horizontal and vertical distances covered by the beam, $x = ct$ and $y = \frac{1}{2}at^2$, giving $y = \frac{a}{2c^2}x^2$. By the equivalence principle, we can then say that light will fall a distance $y = \frac{g}{2c^2}x^2$ in a uniform gravitational field as well. This prediction is even harder to verify than the red-shift phenomenon. A light beam aimed sideways along the earth's surface would fall a distance $\frac{9.8}{2 \cdot 9 \cdot 10^{16}} \cdot 1000^2 = .54 \times 10^{-10}$ meters for every 1000 meters of horizontal travel. This distance is about the radius of a hydrogen atom.

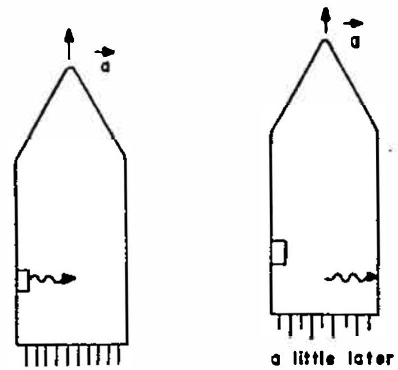

Figure D.3

The deflection of the light of a star around the sun during a total eclipse has been seen, although it is quite small. Since the gravitational field of the sun is not uniform, it is necessary to work out how much light bends around a spherical object. Using his general theory of relativity, Einstein computed the deflection around the sun to be 1.75 seconds of arc. The experiments show a wide degree of scatter, but are at least not in serious disagreement with Einstein's value. General relativity hasn't had the same thorough and wide-ranging experimental verification enjoyed by special relativity, since the experiments in question are so difficult. New experiments are gradually becoming technically feasible, so we can hope for some clarification in the future.

REFERENCE

1. The books by Einstein and several other books listed in the general bibliography contain discussions of the equivalence principle and general relativity.

APPENDIX E
THE TWIN PARADOX

AS MENTIONED IN Chapter IV, the time-dilation effect can lead to apparently paradoxical conclusions. For each of two relatively moving inertial observers to claim that the other observer's clock runs slow seems contradictory and nonsensical. Nevertheless, as illustrated in section D of Chapter VI, there is actually no contradiction. When account is taken of the Lorentz contraction and the fact that two clocks in one frame are not synchronized when viewed from another frame, the "paradox" is resolved. In that section on "Rockets and Clocks," only inertial clocks and inertial observers were considered. No clocks or observers were accelerated during any portion of the experiment. Therefore that discussion has no bearing on the consistency of the twin story, in which one twin departs for Sirius and returns, necessarily accelerating during at least part of the trip. In order to understand this situation from both points of view, it is necessary to know how to deal with accelerating clocks and accelerating observers.

As pointed out several times in the text, special relativity deals only with observations made by inertial observers. Therefore the twin who accelerates cannot use the Lorentz transformation or any deductions from it, such as time dilation or length contraction. This should not be construed as a claim that special relativity can't be used to analyze accelerating objects, including rods and clocks. As long as the acceleration, momentum, energy, and other properties of a particle are measured by inertial observers, the transformations of special relativity can be used to calculate these quantities in any other inertial frame.

Thus our first project in this appendix is a discussion of accelerating clocks, so that the twin who stays at home can calculate the aging of the traveling twin while that twin is accelerating. Secondly, in order to analyze events from the viewpoint of the traveler, we have to know how clocks behave when viewed from accelerating frames of reference.

Neither of these two projects is simple. A thorough analysis of accelerating frames of reference must be reserved for a study of

Einstein's general theory of relativity. We will do here as much as we can with the results of Appendix D, coming from the principle of equivalence. Even the topic of accelerating clocks viewed from inertial frames will involve here a number of assumptions.

With regard to accelerating clocks, we'll assume all our clocks are "ideal." Such a clock is not affected by acceleration per se, but simply runs slow by the time-dilation factor $\sqrt{1 - v^2/c^2}$ appropriate for the velocity it has at each moment. There is probably no real clock which is completely ideal, since an acceleration would likely have some effect on any clock we might devise. As an extreme example, a watch which decelerates as it hits the floor may have its rate drastically altered. Similarly, the traveling twin would not want to accelerate too fast as he leaves for Sirius. An atomic or nuclear clock, whose rate is measured by the frequency of emitted radiation, is a nearly ideal clock for reasonable accelerations. Certainly for the accelerations likely to occur in spaceships, such a clock could be considered ideal. Of course any other kind of clock could be used if it were near enough to ideal, or if it could be corrected for the effects of acceleration.

Assuming that we have such a clock, we would like to know what it reads as compared with a similar clock at rest in an inertial frame. During an infinitesimal time interval dt (as measured by the inertial clock) the accelerating clock will have some velocity v, so will record a time interval $d\tau = dt\sqrt{1 - v^2/c^2}$ due to time dilation. Since the velocity depends on time, this expression has to be integrated for finite time intervals. If we choose $\tau = 0$ when $t = 0$, then the time read by the accelerating clock is

$$\tau = \int_0^t dt\sqrt{1 - v^2/c^2}, \qquad \text{(E-1)}$$

which is called the "integrated proper time."

Regarding accelerating observers and their measurement of clock rates, it was found in Appendix D that clocks run at different rates at different points in an accelerating frame. The effect could be understood as a

gravitational influence on clocks, with those at high altitudes running faster than those at lower altitudes. To first-order accuracy, the ratio of the rates of two clocks is $(1+g\ell/c^2)$, where g is the (uniform) acceleration or gravitational field, and ℓ is the difference in altitude. It is important to note that it is the gravitational potential rather than the gravitational force itself which influences this rate.

We have now assembled the machinery needed to tackle the story of the twins to first-order accuracy from the point of view of both twins. We will find that the relative youthfulness of the traveling twin upon his return is a consistent result, agreed upon by both. An exact analysis confirms this consistency in detail, but is considerably more difficult to carry out, principally because of the need for careful definitions of coordinates and clock rates in accelerating frames. An exact treatment can be found in Møller's book.* The first-order analysis we will do here has the advantage of simplicity and displays the main features of the complete treatment. Throughout the following, all times will refer to the readings of ideal clocks.

The situation is, then, as follows: twin A stays at home, remaining in an inertial frame at all times. Twin B accelerates away, coasts toward Sirius, decelerates to rest, accelerates back, coasts toward the earth, and decelerates and stops, all as shown in Figure E.1. We may as well assume that the trip is symmetrical, so that all the acceleration and deceleration times are of equal duration as read by clocks in A's frame, say $\Delta t_1^{(A)}$, and that the going and coming coast periods have the same velocity and duration $\Delta t_2^{(A)}$. With this introduction we can now calculate the total time interval read by both clocks, first from A's point of view, and then from B's point of view.

To A:

The total time read by A's clock between B's departure and return is

$$\Delta T^{(A)} = 4\Delta t_1^{(A)} + 2\Delta t_2^{(A)}, \qquad \text{(E-2)}$$

* C. Møller. The Theory of Relativity (Oxford University Press, 1952)

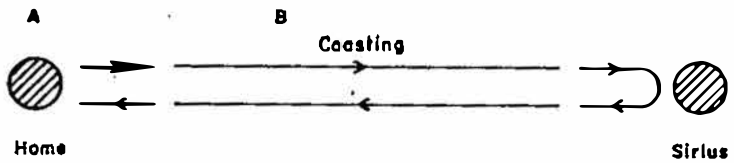

Figure E.1

since there are four acceleration periods and two coasting periods. Now the ratio $\Delta t_1^{(A)}/\Delta t_2^{(A)}$ can be made arbitrarily small, either by increasing the accelerations or by taking a longer trip. That is, we can assume that the acceleration times are negligible compared to the time for the whole trip, and therefore $\Delta T^{(A)} \cong 2\Delta t_2^{(A)}$.

What will B's clock read when he returns? During the coasting periods B's clock will run slow as seen by A, so by the usual time-dilation formula,

$$\Delta t_2^{(B)} = \Delta t_2^{(A)} \sqrt{1 - v^2/c^2}. \tag{E-3}$$

During an accelerating period, B's clock will record a time interval

$$\Delta t_1^{(B)} = \int_0^{\Delta t_1^{(A)}} dt \sqrt{1 - v^2/c^2}. \tag{E-4}$$

Clearly, $\Delta t_1^{(B)} < \Delta t_1^{(A)}$ since the integrand is less than unity over the entire interval. That is, B's clock will run slow while accelerating, but by a varying rate. Since we have already assumed that $\Delta t_1^{(A)}$ is negligible compared to $\Delta t_2^{(A)}$, it follows that $\Delta t_1^{(B)} << \Delta t_2^{(A)}$ also. If in addition the coasting velocity is not so high that $\Delta t_2^{(B)}$ is also negligible compared to $\Delta t_2^{(A)}$, we can say that $\Delta t_1^{(B)} << \Delta t_2^{(B)}$, and so neglect the periods of acceleration for B's clock also. The final result, therefore, is that from A's point of view, B's clock reads less than A's when they reunite. In fact,

$$\Delta T^{(A)} = \frac{\Delta T^{(B)}}{\sqrt{1 - v^2/c^2}} \tag{E-5}$$
$$= \Delta T^{(B)} \left(1 + \frac{v^2}{2c^2} + \ldots\right)$$

to first order in v^2/c^2, using the binomial expansion.

<u>To B:</u>

The more difficult job remains of discovering why B returns younger than A as seen by B <u>himself!</u> During the coasting periods, while B is at rest in an inertial frame, he is allowed to use special relativity. So during this time, he must find that A's clock runs slow, which would seem to imply that A will be younger than B when the trip is over. Therefore the periods of acceleration must be <u>crucial</u> from B's standpoint, because A must age so fast during these periods that he not only overcomes his slower aging during coasting, but actually ages enough extra to make him older than B at the end, by the factor already found. This miracle is wrought by the gravitational influence on clock-rates derived in Appendix D. The magnitude of the effect is so large during the time that B turns around at Sirius, that this period of acceleration is <u>not</u> negligible from B's point of view.

While B is coasting, time dilation of A's clock gives

$$\Delta t_2^{(A)} = \Delta t_2^{(B)} \sqrt{1 - v^2/c^2} = \Delta t_2^{(B)} (1 - v^2/2c^2 + \ldots) \tag{E-6}$$

to first order in v^2/c^2. While B is accelerating, he feels an effective gravitational field in his frame, which will cause clocks A and B to run at different rates. The clock at "higher altitude" will run faster than the clock at "lower altitude" by the factor $(1+g\ell/c^2)$, where g is the acceleration and ℓ is the distance between them (as measured by B). During the initial and final periods of acceleration, A is at "lower altitudes" than B, so will run slower. The factor $(1+g\ell/c^2)$ will vary continuously, since ℓ changes. But for not too high velocities, $v = \sqrt{2g\ell}$ for uniform acceleration, so $g\ell/c^2 \cong v^2/2c^2$, and $(1+g\ell/c^2)$ will vary from unity to something less than three halves. The acceleration time for B is assumed to be negligible compared to the coasting time, so the same is true for A's clock, since it will read somewhat less than B's clock at the end of the acceleration period.

The situation is quite different, however, when B turns around at the midpoint of his journey. Now it is A who is at a "higher altitude," in fact higher by the distance between the earth and Sirius. In terms of B's total travel time and the velocity while coasting, this distance is $\ell = \frac{\Delta T^{(B)}}{2} v$. The acceleration g is the change in velocity divided by the time interval for the acceleration as measured by B, or $2v/\Delta t^{(B)}$. The change in velocity is 2v, because the velocity is first one way and then the other way after the turn-around is completed. Altogether, A runs faster than B by the factor

$$\left(1 + \frac{g\ell}{c^2}\right) = 1 + \frac{2v}{\Delta t^{(B)}} \frac{\Delta T^{(B)} v}{2c^2} = 1 + \frac{v^2}{c^2} \frac{\Delta T^{(B)}}{\Delta t^{(B)}} . \qquad \text{(E-7)}$$

So A will advance an amount

$$\Delta t^{(A)} = \Delta t^{(B)} \left(1 + \frac{v^2}{c^2} \frac{\Delta T^{(B)}}{\Delta t^{(B)}}\right)$$
$$= \Delta t^{(B)} + \frac{v^2}{c^2} \Delta T^{(B)} . \qquad \text{(E-8)}$$

The amazing thing about this result is that no matter how fast B turns around ($\Delta t^{(B)} \to 0$), he will see A age by at least $(v^2/c^2)\Delta T^{(B)}$, which is proportional to the time for the entire trip! A is at such high altitudes in B's effective gravitational field that he ages a great deal almost instantaneously from B's point of view. If B were watching through a telescope, A's hair might turn white in a few seconds!

The total time for the trip to B is $\Delta T^{(B)}$, and the total time for A can be found by adding his aging during the turn-around phase to that from the constant-velocity phase. As before, we take $\Delta t^{(B)} \to 0$ relative to

$\Delta t_2^{(B)}$, so $\Delta T^{(A)} = \Delta T^{(B)} (1 - v^2/2c^2 + \ldots)$ (constant velocity part)

$$+ \frac{v^2}{c^2} \Delta T^{(B)} \qquad \text{(accelerating part)}$$

$$= \Delta T^{(B)} \left(1 + \frac{v^2}{2c^2} + \ldots\right) \qquad \text{(E-9)}$$

which shows that A is older than B at the end of the trip by the same amount (at least to this order in v^2/c^2) as we found before from A's point of view! The new element we've introduced is the gravitational effect on clocks, which must be included when using accelerated frames of reference.

The purpose of the foregoing was to demonstrate the way in which the twin "paradox" can be resolved from the traveling twin's point of view. A number of implicit as well as explicit assumptions have been made. As stated before, a thorough investigation of accelerating frames of reference is both necessary and worthwhile, and leads into the fascinating subject of general relativity.

The twin "paradox" is really no paradox at all. The idea that a contradiction is involved arose from a misunderstanding of the special theory of relativity — namely, that the time-dilation formula could be used in all situations. Since it was derived for inertial frames, it is not at all surprising that it doesn't work for the accelerating twin. The spaceman who goes off to Sirius as in our example really will be younger than his twin brother when he returns.